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ON SOME CRITERION OF CONVERGENCE IN PROBABILITY

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Abstract. Let $(\Omega, \mathscr{A}, P)$ be a probability space. (S, ϱ) denotes a metric space, and $\mathscr{B}$ stands for the σ-field generated by open sets of S. The set S is assumed to be a separable and complete space. A sequence $\{X_n, n \geqslant 1\}$ of random elements, defined on a probability space $(\Omega, \mathscr{A}, P)$ taking values in S, is called *stable* if for every $B \in \mathscr{A}$, with $P(B) > 0$, there exists a probability measure μ_B such that

$$\lim_{n\to\infty} P([X_n \in A]|B) = \mu_B(A).$$

There are given conditions concerning the set $\mathscr{P}_{\mathscr{A}}(S) = \{\mu_B, B \in \mathscr{A}\}$ of probability measures, under which there exists a random element X such that the sequence $\{X_n, n \geqslant 1\}$ of random elements converges in probability to X.

Let $\mathscr{X}$ be the set of all random elements (r.e.):

$$\mathscr{X} = \{X: \Omega \to S;\ X^{-1}(A) \in \mathscr{A},\ A \in \mathscr{B}\}.$$

By $P_X(A) = P([X \in A])$, $A \in \mathscr{B}$, we denote the distribution function of r.e. X. Let $\mathscr{A}_+ = \{B \in \mathscr{A}:\ P(B) > 0\}$ and

$$A^\delta = \{x:\ d(x, A) = \inf_{y\in A} \varrho(x, y) < \delta\}.$$

On the set $\mathscr{P}(S)$ of probability measures, defined on $(S, \mathscr{B})$,

$$L(\tau, \nu) = \inf\{\varepsilon > 0:\ \nu(A) \leqslant \tau(A^\varepsilon)+\varepsilon \text{ and } \tau(A) \leqslant \nu(A^\varepsilon)+\varepsilon,\ A \in \mathscr{B}\} \tag{1}$$

denotes the Lévy-Prohorov metric, where $\tau, \nu \in \mathscr{P}(S)$. Convergence in this metric and weak convergence coincide.

Let

$$r(X, Y) = \inf\{\varepsilon > 0:\ P[\varrho(X, Y) > \varepsilon] < \varepsilon\} \tag{2}$$

and

$$r_1(X, Y) = E\frac{\varrho(X, Y)}{1+\varrho(X, Y)}, \tag{3}$$

where $E(\cdot)$ denotes the mean value, be two metrics introduced in the space $\mathscr{X}$. Convergences with respect to r and r_1 are equivalent to each other and to the convergence in probability ($X_n \xrightarrow{P} X$, $n \to \infty$) [3]. It is known [2] that $L(P_X, P_Y) \leqslant r(X, Y)$. Hence the convergence in probability implies the weak convergence.

Definition 1. A sequence $\{X_n, n \geqslant 1\}$ of r.e. is called *stable* if, for every $B \in \mathscr{A}_+$, there exists a probability measure μ_B such that

$$\lim_{n\to\infty} P([X_n \in A] | B) = \mu_B(A) \quad \text{for every } A \in \mathscr{C}_{\mu_B} = \{A \in \mathscr{B}: \mu_B(\partial A) = 0\},$$

where ∂A denotes the boundary of A and $P(D|B) = P(D \cap B)/P(B)$. In what follows we suppose that $P(A|B) \equiv 0$ and $\mu_B(A) \equiv 0$, whenever $P(B) = 0$, $B \in \mathscr{A}$.

In the special case, where $\mu_B(A) = \mu(A)$ for every $B \in \mathscr{A}_+$, the sequence $\{X_n, n \geqslant 1\}$ of r.e. is called *mixing with density* μ. A survey of stable and mixing sequences of r.e. can be found in [1] and [6].

It is well known [2] that $X_n \xrightarrow{P} X$, $n \to \infty$, iff

(4) $$\lim_{n\to\infty} P([X_n \in A] \dot{-} [X \in A]) = 0 \quad \text{for every } A \in \mathscr{C}_{P_X},$$

where $A \dot{-} B$ denotes the symmetric difference of A and B.

On can prove (cf. [4], [8]) that

(5) $$X_n \xrightarrow{P} X,\ n \to \infty, \text{ iff } L(Q_{X_n}, Q_X) \to 0,\ n \to \infty,$$

for every probability measure Q defined on $(\Omega, \mathscr{A})$ by

$$Q(D) = \big(P(D|B) + P(D)\big)/2, \quad B \in \mathscr{A}_+.$$

LEMMA 1. *If a sequence $\{X_n, n \geqslant 1\}$ of r.e. converges in probability to an r. e. X, then $\{X_n, n \geqslant 1\}$ is stable.*

Proof. If $X_n \xrightarrow{P} X$, $n \to \infty$, then, for every $B \in \mathscr{A}_+$,

$$X_n \xrightarrow{P_B} X,\ n \to \infty, \quad \text{where } P_B(\cdot) = P(\cdot|B).$$

Hence

$$P([X_n \in A]|B) \to P([X \in A]|B), \quad n \to \infty,$$

for every $A \in \mathscr{C}_{P_{X|B}}$, which implies the stability of the sequence $\{X_n, n \geqslant 1\}$ of r.e.

Now we give conditions concerning the set $\mathscr{P}_{\mathscr{A}}(S) = \{\mu_B, B \in \mathscr{A}\}$ of probability measures under which there exists a random element X such that the sequence $\{X_n, n \geqslant 1\}$ of r.e. converges in probability to X.

LEMMA. *Let X and Y be r.e. such that, for all B,*

$$P([X \in A]|B) = P([Y \in A]|B) \quad \textit{for every } A \in \mathscr{B}.$$

Then $X = Y$ almost surely (a.s.).

Proof. If $P([Y \in A]) > 0$, then, by

$$P([X \in A] | [Y \in A]) = P([Y \in A] | [Y \in A]) = 1,$$

we have

$$P([X \in A] \cap [Y \in A]) = P([X \in A]) = P([Y \in A]).$$

Hence $P([X \in A] \dot{-} [Y \in A]) = 0$, which implies that $X = Y$ a.s. as S is a separable space.

For every $X \in \mathscr{X}$ take the set $\mathscr{P}_{\mathscr{A}_+}(S) = \{\mu_B, B \in \mathscr{A}_+\}$ of probability measures defined on $(S, \mathscr{B})$ by

$$\mu_B(A) = P([X \in A] | B), \quad B \in \mathscr{A}_+.$$

It is easy to see that probability measures belonging to $\mathscr{P}_{\mathscr{A}_+}(S)$ satisfy the following conditions:

(I) $P(\bigcup_{k=1}^{n} B_k) \mu_{\bigcup_{k=1}^{n} B_k}(A) = \sum_{k=1}^{n} \mu_{B_k}(A) P(B_k)$ for any $B_1, B_2, \ldots, B_n \in \mathscr{A}$

such that $B_i \cap B_j = \emptyset$, $i \neq j$, $A \in \mathscr{B}$.

(II) If $\mu_B(A) > 0$, then there exists a set $B' \subset B$, $B' \in \mathscr{A}_+$, such that $\mu_{B'}(A) = 1$.

It is not difficult to state that probability measures belonging to $\mathscr{P}_{\mathscr{A}_+}(S)$, satisfying (I), have the following properties:

(6) $$P(\bigcup_{n=1}^{\infty} B_n) \mu_{\bigcup_{n=1}^{\infty} B_n}(A) = \sum_{n=1}^{\infty} \mu_{B_n}(A) P(B_n)$$

for every sequence $\{B_n, n \geqslant 1\}$ of sets such that $B_n \in \mathscr{A}$, $n \geqslant 1$, and $B_i \cap B_j = \emptyset$ when $i \neq j$;

(7) $$\mu_B(A) = 0 \Rightarrow \mu_{B'}(A) = 0 \quad \text{for every } B' \subset B,\ B' \in \mathscr{A}_+,$$
$$\mu_B(A) = 1 \Rightarrow \mu_{B'}(A) = 1 \quad \text{for every } B' \subset B,\ B' \in \mathscr{A}_+;$$

(8) $$(\mu_B(A) = 1 \text{ and } \mu_{B'}(A) = 1) \Rightarrow \mu_{B \cup B'}(A) = 1, \quad B, B' \in \mathscr{A}_+;$$

(9) $$(\mu_{B_n}(A) = 1 \text{ and } B_n \subset B_{n+1}) \Rightarrow \mu_{\bigcup_{n=1}^{\infty} B_n}(A) = 1, \quad B_n \in \mathscr{A}_+.$$

LEMMA 3. *If $\mathscr{P}_{\mathscr{A}_+}(S) = \{\mu_B, B \in \mathscr{A}_+\}$ is a set of probability measures satysfying* (I), (II) *and such that, for a fixed $A \in \mathscr{B}$, $\mu_B(A) > 0$ for some $B \in \mathscr{A}_+$, then there exists a set $D_A(B) \subset B$, $D_A(B) \in \mathscr{A}_+$, such that*

(10) $$\mu_{D_A(B)}(A) = 1,$$

(11) $\mu_C(A) < 1$ *for every* $C \subset B$, $C \in \mathscr{A}_+$, *such that* $P(C \setminus D_A(B)) > 0$,

(12) $\mu_C(A) = 0$ *for every* $C \subset B$, $C \in \mathscr{A}_+$, *such that* $P(C \cap D_A(B)) = 0$,

and

(13) $$\mu_B(A) = P(D_A(B)).$$

Proof. Let

$$\alpha_A = \sup\{P(C)\colon\ C \subset B,\ C \in \mathscr{A}_+ \text{ and } \mu_C(A) = 1\}, \qquad A \in \mathscr{B}.$$

Then there exists a sequence of sets $C_n \in \mathscr{A}_+$, $C_n \subset B$, $n = 1, 2, \ldots$, such that $\mu_{C_n}(A) = 1$ and $P(C_n) \to \alpha_A$, $n \to \infty$. Write

$$C'_n = \bigcup_{k=1}^{n} C_k.$$

Now, by (8), $\mu_{C'_n}(A) = 1$ and, by (9),

$$\mu_{\bigcup_{n=1}^{\infty} C_n}(A) = \mu_{\bigcup_{n=1}^{\infty} C'_n}(A) = 1.$$

Putting $D_A(B) = \bigcup_{n=1}^{\infty} C_n$, we get (10). Moreover, we see that $P(D_A(B)) = \alpha_A$.

To prove (11) assume that $\mu_C(A) = 1$, whenever $P(C \backslash D_A(B)) > 0$, $C \subset B$. Then, by assumption (I), $\mu_{C \backslash D_A(B)}(A) = 1$. Moreover, in view of (8), we have $\mu_{C \cup D_A(B)}(A) = 1$, which with $P(C \backslash D_A B)) > 0$ proves that $P(C \cup D_A(B)) > \alpha_A$ and contradicts the definition of α_A.

To prove (12) assume that $\mu_C(A) > 0$, whenever $P(C \cap D_A(B)) = 0$, $C \subset B$, $C \in \mathscr{A}_+$. By (II) there exists a set $C' \subset C$, $C' \in \mathscr{A}_+$, such that $\mu_{C'}(A) = 1$ and, moreover, $P(C' \cap D_A(B)) = 0$. Hence, by (8) and (10), $\mu_{C \cup D_A(B)}(A) = 1$ and $P(C \cup D_A(B)) > \alpha_A$ as $P(D_A(B)) = \alpha_A$ and $P(C \backslash D_A(B)) > 0$, which contradicts the definition of α_A.

(13) follows from (6), (10) and (12):

$$\begin{aligned}\mu_B(A) &= \mu_{D_A(B)}(A)\,P(D_A(B)) + \mu_{B \backslash D_A(B)}(A)\,P(B \backslash D_A(B)) \\ &= \mu_{D_A(B)}(A)\,P(D_A(B)) = P(D_A(B)),\end{aligned}$$

which completes the proof.

In what follows D_A stands for $D_A(\Omega)$.

LEMMA 4. *Let $\mathscr{P}_{\mathscr{A}_+}(S)$ be a set of probability measures of Lemma 3. Suppose that $\{A_i,\ i \geqslant 1\}$ is a sequence of sets such that $A_i \in \mathscr{B}$, and $\mu_\Omega(A_i) > 0$, $i \geqslant 1$. Then there exists a sequence $\{D_{A_i},\ i \geqslant 1\}$ such that $D_{A_i} \in \mathscr{A}_+$, $i \geqslant 1$, and the following conditions hold*:

(a) $$\mu_\Omega(A_i \cap A_j) = 0 \Rightarrow P(D_{A_i} \cap D_{A_j}) = 0,$$

(b) $$\mu_\Omega(A_i \backslash A_j) = 0 \Rightarrow P(D_{A_i} \backslash D_{A_j}) = 0,$$

(c) *if* $\mu_\Omega(\bigcup_{i=1}^{\infty} A_i) = 1$ *for* $A_i \cap A_j = \emptyset$ *for* $i \neq j$, *then* $P(\bigcup_{i=1}^{\infty} D_{A_i}) = 1$.

Proof. (a) From Lemma 3 we conclude that there exist D_{A_i} and D_{A_j} such that $\mu_{D_{A_i}}(A_i) = 1$ and $\mu_{D_{A_j}}(A_j) = 1$. By the assumption $\mu_\Omega(A_i \cap A_j) = 0$ and (7) we have $\mu_{D_{A_j}}(A_i \cap A_j) = 0$, whence $\mu_{D_{A_j}}(A_i) = 0$. Using once more (7) we conclude that $\mu_{D_{A_i} \cap D_{A_j}}(A_i) = 1$ and $\mu_{D_{A_i} \cap D_{A_j}}(A_i) = 0$. Therefore $D_{A_i} \cap D_{A_j} \notin \mathscr{A}_+$, which proves that $P(D_{A_i} \cap D_{A_j}) = 0$.

(b) follows from (a).

(c) By (13) and (a) we have

$$1 = \mu_\Omega(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu_\Omega(A_i) = \sum_{i=1}^{\infty} P(D_{A_i}) = P(\bigcup_{i=1}^{\infty} D_{A_i}).$$

THEOREM 1. *Let $\mathscr{P}_{\mathscr{A}_+}(S) = \{\mu_B, B \in \mathscr{A}_+\}$ be a set of probability measures. If* (I) *and* (II) *are fulfield, then there exists an r.e. X such that*

$$\mu_B(A) = P([X \in A] | B).$$

Proof. Let $\{A_{i_1, i_2, \ldots, i_k}; i_s \in N, s = 1, 2, \ldots, k\}$ be the class of Borel subsets of S satisfying the following conditions:

(W$_1$) $\quad A_{i_1, i_2, \ldots, i_k} \cap A_{i'_1, i'_2, \ldots, i'_k} = \emptyset \quad$ for $i_k \neq i'_k$;

(W$_2$) $\quad \bigcup_{i_k=1}^{\infty} A_{i_1, i_2, \ldots, i_k} = A_{i_1, i_2, \ldots, i_{k-1}}, \quad \bigcup_{i_1=1}^{\infty} A_{i_1} = S$;

(W$_3$) $\quad d(A_{i_1, i_2, \ldots, i_k}) \leqslant 1/2^k, \quad$ where $d(A) = \sup\{\varrho(x, y)\colon x, y \in A\}$;

(W$_4$) $\quad \mu_\Omega(\partial A_{i_1, i_2, \ldots, i_k}) = 0 \quad$ (cf. 7).

From every set $A_{i_1, i_2, \ldots, i_k}$ we can choose an element $x_{i_1, i_2, \ldots, i_k}$ and define r.e. X_k by the formula

(14) $\quad X_k(\omega) = x_{i_1, i_2, \ldots, i_k} \quad$ for $\omega \in D_{A_{i_1, i_2, \ldots, i_k}} := D_{i_1, i_2, \ldots, i_k}$.

The definition of X_k is correct on the basis of Lemmas 3 and 4.

Using the assumptions we see that

$$\varrho(X_k, X_m) \leqslant 1/2^k \quad \text{a.s. for } m \geqslant k.$$

Because the metric space is complete, there exists an r.e. X such that

$$X_k \to X \text{ a.s.}, \quad k \to \infty.$$

We now prove that

(15) $\quad \mu_B(A) = P([X \in A] | B) \quad$ for $B \in \mathscr{A}_+$ and $A \in \mathscr{B}$.

First we show that (15) holds for $A = A_{i_1, i_2, \ldots, i_k}$ and $B = D_{i_1, i_2, \ldots, i_s} \in \mathscr{A}_+$. The sets $A_{i_1, i_2, \ldots, i_k}$ are continuity sets of the measure μ_Ω, i.e. $A_{i_1, i_2, \ldots, k} \in \mathscr{C}_{\mu_\Omega}$.

If $A \in \mathscr{C}_{\mu_\Omega}$, then for every $\varepsilon > 0$ there exists an n_0 such that

$$\mu_\Omega((\partial A)^{1/2^{n_0}}) < \varepsilon.$$

Let $K = \{i_1, i_2, \ldots, i_{n_0+2};\ A_{i_1,i_2,\ldots,i_{n_0+2}} \cap (\partial A)^{1/2^{n_0+2}} \neq \emptyset\}$.
Then

$$\partial A \subset \bigcup_{i_1,i_2,\ldots,i_{n_0+2}\in K} A_{i_1,i_2,\ldots,i_{n_0+2}} \subset (\partial A)^{1/2^{n_0}},$$

$$[X_{n_0+2} \in \bigcup_{i_1,i_2,\ldots,i_{n_0+2}\in K} A_{i_1,i_2,\ldots,i_{n_0+2}}] = \bigcup_{i_1,i_2,\ldots,i_{n_0+2}\in K} D_{i_1,i_2,\ldots,i_{n_0+2}}$$

and

$$P([X\in\partial A] \cap [X_{n_0+2} \in \bigcup_{i_1,i_2,\ldots,i_{n_0+2}\in K} A_{i_1,i_2,\ldots,i_{n_0+2}}]) = P([X\in\partial A]).$$

Hence

$$P([X\in\partial A]) \leqslant P(\bigcup_{i_1,i_2,\ldots,i_{n_0+2}\in K} D_{i_1,i_2,\ldots,i_{n_0+2}})$$

$$= \mu_\Omega(\bigcup_{i_1,i_2,\ldots,i_{n_0+2}\in K} A_{i_1,i_2,\ldots,i_{n_0+2}}) \leqslant \mu_\Omega((\partial A)^{1/2^{n_0}}),$$

which proves that $A \in \mathscr{C}_{P_X}$. Therefore, we have $A_{i_1,i_2,\ldots,i_k} \in \mathscr{C}_{P_X}$.

Using properties of measure μ_B we see that, for $s \geqslant k$ and any $B \in \mathscr{A}_+$,

$$\mu_{D_{i'_1,i'_2,\ldots,i'_s}\cap B}(A_{i_1,i_2,\ldots,i_k})$$

$$= \begin{cases} 1 \text{ if } i'_l = i_l\ (l = 1, 2, \ldots, k) \text{ and } D_{i'_1,i'_2,\ldots,i'_s} \cap B \in \mathscr{A}_+; \\ 0 \text{ if, for some } 0 \leqslant l \leqslant k,\ i'_l \neq i_l \text{ or } D_{i'_1,i'_2,\ldots,i'_s} \cap B \notin \mathscr{A}_+ \end{cases}$$

and

$$P([X\in A_{i_1,i_2,\ldots,i_k}] | D_{i'_1,i'_2,\ldots,i'_s} \cap B) = \lim_{n\to\infty} P([X_n \in A_{i_1,i_2,\ldots,i_k}] | D_{i'_1,i'_2,\ldots,i'_s} \cap B)$$

$$= P(D_{i_1,i_2,\ldots,i_k} | D_{i'_1,i'_2,\ldots,i'_s} \cap B)$$

$$= \mu_{D_{i'_1,i'_2,\ldots,i'_s}\cap B}(A_{i_1,i_2,\ldots,i_k}).$$

Now, by (6), for any $B \in \mathscr{A}_+$ we have

$$(16)\quad P(B)\,\mu_B(A_{i_1,i_2,\ldots,i_k})$$

$$= P(B)\,\mu_{B\cap \bigcup_{i'_1,i'_2,\ldots,i'_k} D_{i'_1,i'_2,\ldots,i'_k}}(A_{i_1,i_2,\ldots,i_k})$$

$$= \sum_{i'_1,i'_2,\ldots,i'_k} P([X\in A_{i_1,i_2,\ldots,i_k}] | B\cap D_{i'_1,i'_2,\ldots,i'_k})\, P(B\cap D_{i'_1,i'_2,\ldots,i'_k})$$

$$= P([X\in A_{i_1,i_2,\ldots,i_k}] | B)\, P(B).$$

Let F be a closed subset of S and

$$A_n = \bigcup_{\{i_1,i_2,\ldots,i_n: A_{i_1,i_2,\ldots,i_n}\cap F \neq \emptyset\}} A_{i_1,i_2,\ldots,i_n}.$$

It is obvious that $A_n \supset A_{n+1} \supset F$ for $n = 1, 2, \ldots$ and $\bigcap_{n=1}^{\infty} A_n = F$. Hence by the continuity axiom and (15)

$$\mu_B(F) = \lim_{n\to\infty} \mu_B(A_n) = \lim_{n\to\infty} P([X\in A_n]|B) = P([X\in F]|B)$$

and, by well known property of measure,

$$\mu_B(A) = P([X\in A]|B) \quad \text{for every } A\in\mathscr{B}.$$

THEOREM 2. *Let* $\{X_n; n \geqslant 1\}$ *be a stable sequence of r.e. and*

(17) $$\lim_{n\to\infty} P([X_n\in A]|B) = \mu_B(A) \quad \textit{for } B\in\mathscr{A}_+ \textit{ and } A\in\mathscr{C}_{\mu_B}.$$

If the measures μ_B, $B\in\mathscr{A}_+$, *satisfy condition* (II), *then there exists an r.e.* X *such that* $X_n \xrightarrow{P} X$, $n\to\infty$.

Proof. It is easy to see that the measure μ_B, $B\in\mathscr{A}_+$, satisfy condition (I). By Theorem 1 and (17) there exists an r.e. X such that

$$\lim_{n\to\infty} P([X_n\in A]|B) = P([X\in A]|B), \quad B\in\mathscr{A}_+,\ A\in\mathscr{C}_{P_X}.$$

Hence $\lim_{n\to\infty} L(Q_{X_n}, Q_X) = 0$ for every measure Q defined by

$$Q(D) = [P(DB)+P(D)]/2, \quad B\in\mathscr{A}_+.$$

By (5), $X_n \xrightarrow{P} X$, $n\to\infty$, which completes the proof of Theorem 2.

Let $Q_A(B) = \mu_B(A)P(B)$. It is well known that $Q_A(\cdot)$ is absolutely continuous measure with respect to P and

$$Q_A(B) = \int_B \alpha_A\, dP,$$

where α_A denotes density of sequence $\{X_n, n\geqslant 1\}$.

As a consequence of Theorem 2 we have

THEOREM 3. *A stable sequence* $\{X_n, n\geqslant 1\}$ *of r.e. converges in probability to an r.e.* X *iff*

$$\alpha_A(\omega) = \begin{cases} 1 & \textit{for } \omega\in D_A, \\ 0 & \textit{for } \omega\notin D_A \end{cases}$$

for every $A\in\mathscr{B}$.

Proof. Let, for every $A\in\mathscr{B}$,

$$\alpha_A(\omega) = \begin{cases} 1 & \text{for } \omega\in D_A, \\ 0 & \text{for } \omega\notin D_A. \end{cases}$$

Then

$$Q_A(B) = \int_B \alpha_A\, dP = P(D_A\cap B) = P(D_A|B)P(B) := \mu_B(A)P(B).$$

μ_B satisfies (II). Indeed, if $0 < \mu_B(A) = P(D_A|B)$, then there exists a subset $B' = D_A \cap B$ of B such that

$$\mu_{B'}(A) = P(D_A|B') = 1.$$

Moreover, we know that (I) is satisfied if $\{X_n, n \geqslant 1\}$ is stable. Therefore, by Theorem 2, $X_n \xrightarrow{P} X$, $n \to \infty$.

Assume now that, for some $A \in \mathscr{B}$, $B_0 = \{\omega: 0 < \alpha_A(\omega) < 1\}$ and $P(B_0) > 0$. Then, for every $B \subset B_0$, $B \in \mathscr{A}_+$, we have

$$0 < Q_A(B_0) = \int_{B_0} \alpha_A \, dP < P(B_0)$$

and

$$0 < Q_A(B) = \mu_B(A) P(B) = \int_B \alpha_A \, dP < P(B),$$

which proves that $\mu_B(A) < 1$. Now the assumption that $X_n \xrightarrow{P} X$, $n \to \infty$, leads to the contradiction condition (II). This completes the proof of Theorem 3.

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REFERENCES

[1] D. J. Aldous and G. K. Eagleson, *On mixing and stability of limit theorems*, Ann. Probability 6 (1978), p. 325-331.

[2] P. Billingsley, *Convergence of probability measures*, New York 1968.

[3] D. Dugue, *Statistique théorique et appliguée*, Masson et C^{ie}, Paris 1958.

[4] P. Fernandez, *A note on convergence in probability*, Boletim Soc. Bras. Mat. 3 (1972), p. 13-16.

[5] R. Fischler, *Stable sequences of random variables and the weak convergence of the associated empirical measures*, Sankhya, A 33 (1971), p. 67-72.

[6] A. Renyi, *On stable sequences of events*, Sankhya, A 25 (1963), p. 293-302.

[7] A. V. Skorohod, *Limit theorems for stochastic processes*, Theor. Probability Appl. 1 (1956), p. 289-319.

[8] D. Szynal and W. Zięba, *On some type of convergence in law*, Bull. Acad. Polon. Sci. (1974), p. 1143-1149.

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